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Linear algebra – long test – solutions

Problem 1. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

Find the matrix $A^{-1}B$.

Solution I. First, we find A^{-1} by Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 5 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 & -3 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 5 & 6 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & 5 & 6 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

Thus,

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ 5 & 6 & -2 \\ 0 & 1 & 0 \end{bmatrix}, \ A^{-1}B = \begin{bmatrix} -2 & -3 & 1 \\ 5 & 6 & -2 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -7 & -4 \\ 12 & 17 & 10 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution II. (don't read it if you haven't mastered the first method) It is possible to find $A^{-1}B$ without computing A^{-1}. To this end, we solve the system

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 5 & 2 & 3 & 2 & 2 & 3 \end{array}\right]$$

with A on the left and B on the right. Performing Gaussian elimination, we arrive at

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -4 \\ 0 & 1 & 0 & 12 & 17 & 10 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right],$$

with I on the left and $A^{-1}B$ on the right.

This requires some justification. Recall that in each step of Gaussian elimination, both matrices get multiplied from the left by some other matrix (*matrix of elementary operation*). At the end, we have $E \cdot A$ on the left and $E \cdot B$ on the right. But since $E \cdot A = I$, we necessarily have $E = A^{-1}$ and so we have our answer on the right.

Problem 2. Let $\mathcal{A} = \{(2,0,5),(1,0,2),(0,1,3)\}$ and $\mathcal{B} = \{(2,1,2),(3,1,2),(2,1,3)\}$ be two bases of $\mathbb{R}^3$, and let $\mathcal{C}$ be a basis of $\mathbb{R}^2$. The linear map $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is given by the matrix

$$[\varphi]_{\mathcal{A}}^{\mathcal{C}} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix}.$$

- Write a formula for the matrix $[\varphi]_{\mathcal{B}}^{\mathcal{C}}$ in terms of some of the matrices $[\varphi]_{\mathcal{A}}^{\mathcal{C}}$, $[\mathrm{id}]_{\mathrm{st}}^{\mathcal{A}}$, $[\mathrm{id}]_{\mathcal{A}}^{\mathrm{st}}$, $[\mathrm{id}]_{\mathrm{st}}^{\mathcal{B}}$, $[\mathrm{id}]_{\mathcal{B}}^{\mathrm{st}}$. *Hint.* Consider the composition $\varphi \circ \mathrm{id} \circ \mathrm{id}$.
- Find the matrix $[\varphi]_{\mathcal{B}}^{\mathcal{C}}$. *If you wish, you may do it without the formula required above.*

Using the formula for the composition $\varphi \circ \mathrm{id} \circ \mathrm{id}$ with bases $\mathcal{B}, \mathrm{st}, \mathcal{A}, \mathcal{C}$, we obtain:

$$[\varphi]_{\mathcal{B}}^{\mathcal{C}} = [\varphi]_{\mathcal{A}}^{\mathcal{C}} \cdot [\mathrm{id}]_{\mathrm{st}}^{\mathcal{A}} \cdot [\mathrm{id}]_{\mathcal{B}}^{\mathrm{st}}.$$

To find $[\varphi]_{\mathcal{B}}^{\mathcal{C}}$, we will need $[\mathrm{id}]_{\mathrm{st}}^{\mathcal{A}}$ and $[\mathrm{id}]_{\mathcal{B}}^{\mathrm{st}}$. To obtain $[\mathrm{id}]_{\mathcal{B}}^{\mathrm{st}}$, we put the vectors from the basis $\mathcal{B}$ as columns; let us observe that this gives us the matrix B from the previous problem. Similarly, $[\mathrm{id}]_{\mathcal{A}}^{\mathrm{st}}$ is equal to A, and $[\mathrm{id}]_{\mathrm{st}}^{\mathcal{A}}$ is its inverse, A^{-1}. Employing the calculations from the previous problem and using the formula above, we have

$$[\mathrm{id}]_{\mathrm{st}}^{\mathcal{A}} \cdot [\mathrm{id}]_{\mathcal{B}}^{\mathrm{st}} = A^{-1} \cdot B = \begin{bmatrix} -5 & -7 & -4 \\ 12 & 17 & 10 \\ 1 & 1 & 1 \end{bmatrix},$$

$$[\varphi]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -5 & -7 & -4 \\ 12 & 17 & 10 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ -12 & -18 & -9 \end{bmatrix}.$$

Problem 3. Compute the determinants:

$$\det \begin{bmatrix} -5 & -7 & 3 & 3 \\ 3 & 4 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}, \quad \det \begin{bmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

To compute the first determinant, we substract the 4th column from the 3rd, use the Laplace expansion with respect to the 3rd column, then subtract 3 times the 3rd row from the 1st, use the Laplace expansion with respect to the 3rd column, and finally use the formula for 2×2 matrices.

$$\det \begin{bmatrix} -5 & -7 & 3 & 3 \\ 3 & 4 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} -5 & -7 & 0 & 3 \\ 3 & 4 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} -5 & -7 & 3 \\ 3 & 4 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \det \begin{bmatrix} -5 & -13 & 0 \\ 3 & 4 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} -5 & -13 \\ 3 & 4 \end{bmatrix} = (-5) \cdot 4 - (-13) \cdot 3 = -20 + 39 = 19$$

Each time we use the Laplace expansion, there is only one nonzero element in the column. Luckily, it equals 1 and the sign is positive, so the calculations are more tidy than usual.

For the second determinant, we subtract 2 times the 2nd row from other rows, subtract these two rows from the 2nd row, swap rows to get an upper-triangular matrix, then use the formula for such matrices.

$$\det \begin{bmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -(1 \cdot 1 \cdot 1) = -1$$